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Model for Energy Transfer in Isotropic Turbulence

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ABSTRACT

An approximate energy-transfer function for isotropic turbulence is proposed on the basis of an analogy with radiative transfer in an inhomogeneous medium. An essential feature of the approximation is replacement of the actual triad interactions of the Fourier modes by interactions between pairs of modes. The interaction of each pair of modes satisfies detailed conservation of energy. Regardless of which mode has the higher wavenumber, the net transfer of energy is always from the more strongly excited to the more weakly excited mode of a pair. The transfer function gives an inertial range where the spectrum obeys the Kolmogorov law and where energy transfer is by local cascade. The spectrum in the far dissipation range falls off sufficiently rapidly that all spatial derivatives of the velocity field exist in mean square.

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1. INTRODUCTION

One reason for studying isotropic turbulence is the hope that there will emerge simple models of nonlinear processes which can be generalized and applied to turbulent flows that occur in nature. A nonlinear process of central importance is the transfer of energy among the spatial Fourier modes of the velocity field. Only limited process has been made in the analytical study of turbulent energy-transfer. In the last fifteen years, however, a number of elementary physical models for this process have been proposed.[1-4]

The purpose of the present paper is to add still another elementary energy-transfer model to the list. Our justification is a hope that the new model may prove more suitable than existing ones for generalization and application to certain physical problems. One application we envisage is thermally driven turbulence, wherein it is of interest to determine the velocity spectrum at wavenumbers below, as well as above, the wavenumbers at which energy is fed into the turbulence. In this problem it is inadequate to represent the energy-transfer by a one-way cascade toward high wavenumbers as is done, for example, in the Heisenberg transfer model.[5]

The model we propose is based on an analogy between turbulent energy transfer and radiative transfer in an inhomogeneous medium.[6] This analogy is suggested by some qualitative consequences of the previously proposed "direct-interaction" approximation for turbulent energy transfer.[7] However, the present approximation is much simpler than the direct-interaction

approximation, and very little mathematical analysis is required to obtain it. An essential feature of the simplification is that the interactions among triads of Fourier modes, which characterize both the exact equations of turbulence and the direct-interaction approximation, are replaced by interactions between pairs of modes. This feature is present in Heisenberg's transfer model also.[1] As we shall point out, a penalty incurred for our misrepresentation of the actual dynamics is a significant loss of naturalness in portraying the interaction among distant wavenumbers.

In Sec. 2, we shall state some important qualitative properties of the exact transfer process which are deducible analytically or strongly indicated by experimental evidence and physical intuition. In the following sections we shall describe the analogy to radiative transfer, introduce the proposed transfer approximation, and discuss the extent to which the approximation reproduces the properties stated in Sec. 2.

2. PROPERTIES OF THE EXACT TRANSFER PROCESS

The energy-balance equation for incompressible isotropic turbulence obeying the Navier-Stokes equation may be written in the form[8]

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) E(k) = T(k). \qquad (2.1)$$

Here ν is the kinematic viscosity, $E(k)$ is the usual wavenumber spectrum of kinetic energy [energy-per-unit-mass = $\int_0^\infty E(k)dk$], and $T(k)dk$ is the net rate of transfer of energy-per-unit-mass from all other wavenumbers

to the wavenumbers in the range dk. Let us consider isotropic turbulence as the limit $L \to \infty$ of turbulence contained in a cubical box of side L. Then the exact form of T(k) is

$$T(k) = \lim_{L \to \infty} 4\pi k^2 \left(\frac{L}{2\pi}\right)^{3/2} \mathrm{Im} \iint \delta(\underset{\sim}{p}+\underset{\sim}{q}+\underset{\sim}{k}) \langle \underset{\sim}{k} \cdot \underset{\sim}{u}(\underset{\sim}{p}) \underset{\sim}{u}(\underset{\sim}{q}) \cdot \underset{\sim}{u}(\underset{\sim}{k}) \rangle d^3p d^3q, \quad (2.2)$$

where $\underset{\sim}{u}(\underset{\sim}{k})$ is a suitably normalized Fourier amplitude of the velocity field, $\delta(\underset{\sim}{k})$ is the three-dimensional Dirac function, Im denotes imaginary part, and $\langle\ \rangle$ denotes average over the isotropic ensemble.[9]

Equation (2.2) expresses the fact that the nonlinear interaction is a sum over elementary interactions involving <u>triads</u> of wavevectors. This equation may be rewritten in the form

$$T(k) = \frac{1}{2} \iint_{\Delta} S(k|p,q)\,dpdq, \quad (2.3)$$

where $S(k|p,q)$ has the symmetry property

$$S(k|p,q) = S(k|q,p).$$

Here k, p, and q are wave<u>numbers</u>, and the integration $\iint_{\Delta}$ is over all values of p and q such that k, p, and q can form the legs of a triangle. $S(k|p,q)dpdq$ is the total contribution to T(k) from all wave<u>vectors</u> $\underset{\sim}{p}$ and $\underset{\sim}{q}$ in (2.2) such that p and q lie within given ranges dp and dq, plus the contribution obtained by interchanging $\underset{\sim}{p}$ and $\underset{\sim}{q}$. It is easily shown from (2.2) and the condition of incompressibility that

$$S(k|p,q) + S(p|q,k) + S(q|k,p) = 0. \quad (2.4)$$

That is, the elementary interaction among any three wavenumbers k, p, and q is conservative. From (2.4) there follows the overall conservation property

$$\int_0^\infty T(k)dk = 0. \tag{2.5}$$

There is a further exact property which provides an insight into the nature of the energy transfer, even though the property concerns a hypothetical system rather than an actual turbulent fluid. Suppose that we alter the Navier-Stokes equation, written in terms of the Fourier rerpesentation, by taking $\nu = 0$ and removing from the equation all Fourier modes which lie above some arbitrary cut-off wavenumber. Then it may be shown that equilibrium (time-invariant) distributions exist in which the total energy is equally divided among the Fourier modes which remain.[10-12] That is,

$$E(k) \propto k^2, \tag{2.6}$$

where the factor k^2 appears because the density of Fourier modes between wavenumbers k and k + dk is $\propto k^2$. These equipartition distributions satisfy the condition of detailed balance,

$$S(k|p,q) = 0 \tag{2.7}$$

for all k, p, and q.

The equipartition distribution is far from realized in actual turbulence. Viscous dissipation and the drain of energy to distant

wavenumbers provide sinks which prevent absolute equilibrium from being produced. Nevertheless, the equipartition results suggest that in actual turbulence the energy transfer will in general be from strongly to weakly excited Fourier modes and that $S(k|p,q)$ typically may vanish, or be small, for triads of modes which satisfy

$$k^{-2}E(k) = p^{-2}E(p) = q^{-2}E(q). \qquad (2.8)$$

The remaining properties which we wish to attribute to the actual energy-transfer process represent hypotheses rather than deductions. They are based on empirical evidence and physical intuition. We assume, following Kolmogorov and others, that, at sufficiently high Reynolds numbers, the spectrum consists of well-defined energy-containing, inertial, and dissipation ranges.[8] In the inertial range, we assume that the energy transfer consists of a cascade process, essentially local in wavenumber, by which energy from the energy-containing range reaches the dissipation range. Finally, we assume that the spectrum in the dissipation range eventually falls off rapidly enough that all the spectral moments

$$\int_0^\infty k^{2m}E(k)dk \qquad (m = 1, 2, \ldots) \qquad (2.9)$$

exist. This implies that all spatial derivatives of the velocity field exist in mean square. We base the last assumption on available experimental evidence[8,13] and on the physical intuition that the action of viscosity should prevent the formation of infinitely sharp shear fronts.

In summary, we now list five deduced or presumed properties of the exact transfer process:

(i) The nonlinear interaction is a sum of elementary interactions among triads of Fourier modes.

(ii) The elementary interactions are individually conservative; consequently, the total nonlinear interaction is conservative.

(iii) For the hypothetical system with $\nu = 0$ and a k cut-off, the equipartition spectrum (2.6) yields detailed balance among the Fourier modes that remain.

(iv) At sufficiently high Reynolds numbers, the energy and dissipation ranges are separated by an inertial range in which transfer is by a local cascade process.

(v) The transfer yields a spectrum for which all the moments (2.9) exist.

3. ANALOGY WITH RADIATIVE TRANSFER

In several recent papers,[7,12,14] one of us has investigated turbulent energy transfer on the basis of the so-called direct-interaction approximation. This corresponds to taking a particular partial summation from all orders of an exact expansion of the energy transfer in powers of a characteristic turbulent Reynolds number. No arbitrary parameters or functions are invoked. The approximation exhibits all of the properties (i)-(v) listed at the end of Sec. 2.[7,12]

The direct-interaction approximation is simplest in the case of

stationary, isotropic turbulence.[15] There it yields

$$S(k|p,q) = S_a(k|p,q) - S_e(k|p,q), \tag{3.1}$$

where

$$\begin{aligned} S_a(k|p,q) &= A_{kpq}E(p)E(q), \\ S_e(k|p,q) &= [B_{kpq}E(q) + B_{kqp}E(p)]E(k). \end{aligned} \tag{3.2}$$

The coefficients A_{kpq} and B_{kpq} are given by

$$\begin{aligned} A_{kpq} &= k^3(pq)^{-1}(1 - xyz - 2y^2z^2)\Theta_{kpq}, \\ B_{kpq} &= p^2q^{-1}(xy + z^3)\Theta_{pqk}, \end{aligned} \tag{3.3}$$

where x, y, and z are the cosines of the interior angles opposite k, p, and q, respectively, in a triangle with these wavenumbers as sides. The quantity Θ_{kpq} is a relaxation time of the triple correlation of a triad of modes k, p, and q under the joint action of viscous and dynamic ('eddy') dissipation processes. In many cases, Θ_{kpq} may be approximated over much of the spectrum by

$$\Theta_{kpq} \sim [\nu k^2 + (2/\pi)^{1/2}v_0(k^2 + p^2 + q^2)^{1/2}]^{-1},$$

where v_0 is the rms value of the turbulent velocity in any direction. In general, however, E(k) and Θ_{kpq} must be determined simultaneously by solving a pair of integrodifferential equations.[7]

The quantity A_{kpq} in (3.2) is always positive, and B_{kpq} is typically positive. Thus $S_a(k|p,q)$ may be regarded as an absorption

term, which always represents a flow of energy into wavenumber k from p and q, while $S_e(k|p,q)$ represents an emission of energy from wavenumber k. It will be noted from (3.2) that $S_a(k|p,q)$ involves E(p) and E(q) but not E(k). On the other hand, $S_e(k|p,q)$ is proportional to E(k). The conservation property (2.5) implies that the integral of $S_e(k|p,q)$ over all k, p, and q equals the integral of $S_a(k|p,q)$. That is, all the energy emitted from any wavenumber is absorbed by other wavenumbers.

The facts just stated suggest an analogy between the transfer of turbulent kinetic energy in k space and the transfer, in x space, of electromagnetic energy in an inhomogeneous medium which both radiates and absorbs.[6] If the analogy were perfect, it would be appropriate to write T(k) in the form

$$T(k) = -\mathcal{E}(k) + \int_0^\infty \mathcal{E}(k')f(k,k')dk', \tag{3.4}$$

where $\mathcal{E}(k)dk$ is the total emission from wavenumbers in the range dk and f(k,k')dk gives the fraction of the emission from wavenumbers in the range dk' which is absorbed in the range dk. The condition that all the energy emitted be absorbed somewhere is

$$\int_0^\infty f(k,k')dk = 1. \tag{3.5}$$

Consistency of (3.4) with (3.1) would imply

$$\mathcal{E}(k) = \tfrac{1}{2} \iint_{\Delta} S_e(k|p,q)\,dpdq,$$

$$\int_0^\infty \mathcal{E}(k')f(k,k')\,dk' = \tfrac{1}{2}\iint_{\Delta} S_a(k|p,q)\,dpdq. \qquad (3.6)$$

It is apparent that our analogy is not exact. By appeal to radiative transfer theory, we would expect that $\mathcal{E}(k)$ should depend only on $E(k)$, the strength of excitation at wavenumber k. But this is not supported by (3.2), or by the Navier-Stokes equation itself. Moreover, the form of (3.4) implies interactions between pairs of Fourier modes, a feature which does not correspond to the actual triad nature of the elementary interactions. Hence it would be difficult to determine unambiguously the form of $f(k,k')$ from the second of relations (3.6).

Despite the objections just raised, the authors find (3.4) attractive as a basis for constructing approximations to $T(k)$. The similarity to radiative transfer equations could be expected to suggest both qualitative insights and techniques for computation. Also, the replacement of triad by pair interactions means that double integrals are replaced by single integrals, which may be a valuable feature in numerical calculations. Finally, it is possible to provide a degree of physical motivation for (3.4) on the basis of property (iv) of Sec. 2. We shall now state the argument briefly.

The localness of energy cascade implies that nearby wavenumbers are somehow more strongly coupled than distant ones. This makes it

plausible that the emission from wavenumber k should depend principally on the excitation of wavenumbers whose order of magnitude is k. Also, it seems plausible that the triad interactions most important for transferring energy among wavenumbers that differ substantially in magnitude may be interactions in which two of the three wavenumbers are of the same order of magnitude. Suppose now that we adopt an idealization in which all wavenumbers in some appropriate neighborhoods of two wavenumbers k and k' are regarded as indistinguishable from k and k' themselves. We then come rather naturally to (3.4) and to the assumption that $\varepsilon(k)$ depends only on E(k). Some of the limitations on this argument will be discussed at the end of the next section.

4. PROPOSED TRANSFER APPROXIMATION

The considerations presented in Secs. 2 and 3 have led the authors to the following transfer approximation:

$$T(k) = \int_0^\infty S(k|p)dp, \tag{4.1}$$

where

$$\begin{aligned} S(k|p) &= S_a(k|p) - S_e(k|p), \\ S_a(k|p) &= S_e(p|k), \\ S_e(k|p) &= \eta[k^{-2}E(k)]^{3/2}(kp)^{7/4}g(p/k). \end{aligned} \tag{4.2}$$

Here η is a numerical constant and $g(p/k)$ is a non-negative, dimensionless function that measures the strength of coupling between wavenumbers k and p.

The factor $(kp)^{7/4}$ makes $S_e(k|p)$ dimensionally correct. We impose upon g the symmetry condition and normalization

$$g(k/p) = g(p/k), \tag{4.3}$$

$$\int_0^\infty x^{7/4} g(x)dx = 1. \tag{4.4}$$

These relations imply

$$g(x) < O(x^{-11/4}), \qquad x \to \infty, \tag{4.5}$$

a condition which gives a lower bound to the localness of energy transfer. For the moment, we leave g otherwise undetermined.

The analogy between the present approximation and the direct-interaction approximation becomes immediately apparent if we compare (4.1) and (4.2) with (2.3), (3.1), and (3.2). The most important qualitative difference is that we have replaced triad interactions with pair interactions. Upon performing an integration over $S_e(k|p)$ and using (4.4), we find that (4.1) and (4.2) may be written in the form (3.4), (3.5) with

$$\mathcal{E}(k) = \eta[kE(k)]^{3/2} \tag{4.6}$$

and

$$f(k,p) = p^{-1}(k/p)^{7/4} g(k/p). \tag{4.7}$$

Equation (4.6) represents the unique dimensionally correct form for $\mathcal{E}(k)$ which depends solely on k and E(k).

Let us now ask to what extent the proposed transfer approximation satisfies the requirements (i)-(v) listed at the end of Sec. 2:

(i) It has already been noted that the approximation replaces triad interactions with pair interactions. The rationale of this procedure was discussed in Sec. 3, and it will be taken up again at the end of the present Section.

(ii) There follows immediately from (4.2) the detailed conservation property

$$S(k|p) + S(p|k) = 0, \tag{4.8}$$

which replaces (2.4). The overall conservation condition (2.5) follows from (4.8).

(iii) The detailed balance relation

$$S(k|p) = 0 \qquad \text{(all } k \text{ and } p\text{)} \tag{4.9}$$

follows directly from (4.2) and (4.3) when the spectrum has the equipartition form (2.6). More generally, it is clear from (4.2) and (4.3) that $S(k|p) \geq 0$ if $k^{-2}E(k) < p^{-2}E(p)$. That is, the net transfer is always to the more weakly excited mode of a pair, regardless of which mode has the higher wavenumber.

(iv) We shall verify that, as a consequence of (4.5), the approximation yields an inertial range through which the energy transfer proceeds by local cascade. The spectrum in the inertial range has the Kolmogorov form

$$E(k) = \alpha \epsilon^{2/3} k^{-5/3}, \tag{4.10}$$

where α is a numerical constant, which we shall determine in terms of η, and ϵ is the rate of dissipation of kinetic-energy-per-unit-mass by viscosity.[16]

(v) We shall see that the existence of the spectral moments (2.9) is not automatic but requires that $g(k/p)$ vanish with at least exponential strength as $k/p \to \infty$. This fact, together with considerations of simplicity, has led the authors to propose the choice

$$g(x) = N \exp[-a(x + x^{-1})], \tag{4.11}$$

where a is a positive constant and N is the normalizing factor called for by (4.4). We may either find an optimum value for a by fitting the dissipation-range spectrum to experimental data, or simply take a = 1. It will be noted below that choices $g(x)$ other than (4.11) may be desirable in applications of the transfer approximation.

In the remainder of this Section, we shall examine in some detail the structure of the inertial range and the behavior of $E(k)$ as $k \to \infty$, thereby verifying the statements made under headings (iv) and (v). First, let us substitute (4.10) into (4.2) and (4.1), thereby obtaining

$$T(k) = \eta\alpha^{3/2}\epsilon \int_0^\infty k^{7/4} p^{-15/4} g(k/p)\,dp$$

$$-\eta\alpha^{3/2}\epsilon \int_0^\infty k^{-15/4} p^{7/4} g(p/k)\,dp.$$

If we make the respective substitutions $x = k/p$ and $x = p/k$ in the two

integrals on the right side of this equation, we find, using (4.5), that the integrals are equal and that they converge at both limits. Thus T(k) vanishes, showing the existence of equilibrium under the action of inertial forces alone. The convergence of the integrals at the limits $p = 0$ and $p = \infty$ shows that wavenumbers $p \ll k$ and $p \gg k$ contribute negligibly to T(k). This demonstrates the localness of the energy cascade in the inertial range.

In order to determine the constant α in (4.10), we note that, according to the conservation property, the dissipation ϵ must equal the power passing through the inertial range by cascade. For any wavenumber k, the total net rate of transfer of energy from all wavenumbers $p < k$ to all wavenumber $q > k$ is clearly

$$\Pi(k) = \int_k^\infty dq \int_0^k dp S(q|p). \tag{4.12}$$

Substituting (4.10) into (4.12), setting $\epsilon = \Pi(k)$, and performing appropriate variable changes and partial integrations, we find

$$\begin{aligned} \alpha &= [\eta I]^{-2/3}, \\ I &= \int_1^\infty \left(x^{7/4} - x^{-15/4} \right) g(x)(\ln x) dx. \end{aligned} \tag{4.13}$$

Hence, α is fixed in terms of η once the form of g is prescribed. The two terms which comprise I represent, in non-dimensional form, the total rate of energy-absorption and energy-emission, respectively, by wavenumbers $> k$ due to interactions with modes $< k$.

Now let us consider the existence of the spectral moments (2.9). Suppose that instead of (4.11) we had taken

$$g(k/p) \sim b(k/p)^{-n} \qquad (k \gg p), \tag{4.14}$$

where b is a constant and $n > 11/4$. Consider a wavenumber k in the far tail of the dissipation range. That is, $k \gg k_s$, where k_s is the Kolmogorov microscale $(\epsilon/\nu^3)^{1/4}$. Let us assume, subject to a check for consistency, that the significant contributions to the absorption term $\int_0^\infty S_a(k|p)dp$ in T(k) come from wavenumbers $p \lesssim k_s$. Also, let us assume that the emission term $\int_0^\infty S_e(k|p)dp$ is negligible compared to the absorption term. The second assumption is equivalent to the statement that almost all of the energy absorbed at wavenumber k is dissipated by viscosity rather than re-emitted.

According to the energy-balance equation (2.1), the condition for a steady state at wavenumber k under our assumptions is

$$2\nu k^2 E(k) \sim \eta \int_0^\infty [p^{-2}E(p)]^{3/2}(kp)^{7/4} g(k/p)dp, \tag{4.15}$$

whence, using (4.14),

$$E(k) \sim \tfrac{1}{2}\nu^{-1} b k^{-n-1/4} \int_0^\infty p^{n-5/4}[E(p)]^{3/2}dp. \tag{4.16}$$

The consistency of our first assumption may now be checked by substituting $E(p) \sim E(k_s)(p/k_s)^{-n-1/4}$ in the right side of (4.15) and verifying

that the integral converges as $p/k_s \to \infty$. Our second assumption may be verified in a related fashion.

It is clear from (4.16) that the choice (4.14) is not consistent with the existence of all the moments (2.9), no matter how large n may be. In order for all the moments to exist, $g(k/p)$ must vanish at least exponentially as $k/p \to \infty$. Apart from this requirement, together with (4.3) and (4.4), the form of $g(k/p)$ is undetermined by our considerations so far. We have proposed the form (4.11) principally upon grounds of simplicity. However, this choice has also been influenced by the exponential behavior of $E(k)$ that is predicted by the direct-interaction approximation in the far dissipation range.[7] In any particular application of the transfer approximation, it may be desirable to take a different form for $g(k/p)$ in order to achieve simplicity in computations. The replacement of triad interactions by pair interactions, which is basic to our transfer approximation, is a sufficiently grave mutilation of the exact dynamics that there seems little point in pursuing elaborate theoretical arguments as to what the precise form of $g(k/p)$ should be.[17]

The strong requirement which we have been led to impose on the vanishing of $g(k/p)$ as $k/p \to \infty$ actually represents a rather serious artificiality in the portrayal of the interactions of very distant wavenumbers. The origin lies in the replacement of triad interactions by pair interactions. The coefficients which occur in the k space form of the Navier-Stokes equation are all algebraic, and it is clear that they cannot, in themselves, lead to an exponentially small coupling between distant Fourier amplitudes. However, because of the convolutory character

of the equation, the coupling between any two amplitudes $\underset{\sim}{u}(\underset{\sim}{k})$ and $\underset{\sim}{u}(\underset{\sim}{p})$ involves a third amplitude $\underset{\sim}{u}(\underset{\sim}{q})$ as a modulating factor. If $k \gg p$, then $q \approx k$. If $\underset{\sim}{k}$ is high in the dissipation range, the amplitude $\underset{\sim}{u}(\underset{\sim}{q})$ will therefore be very small, and this accounts for the smallness of the effective coupling of $\underset{\sim}{k}$ and $\underset{\sim}{p}$. In our transfer approximation, this phenomenon, which in actuality depends on the fact that the spectrum falls off very rapidly in the dissipation range, is artificially represented by a fixed function $g(k/p)$ which is independent of the spectrum.

In the direct-interaction approximation (3.1)-(3.3), the coefficients A_{kpq} and B_{kpq}, which are analogous to $g(k/p)$, behave algebraically for $k \gg p$, and the exponential fall-off of the dissipation-range spectrum[7] is due to modulation effects which resemble those in the exact dynamics. A consequence is that the energy-dynamics of the far-dissipation range differ considerably in the direct-interaction approximation and our present approximation, even if $g(k/p)$ is chosen to make the asymptotic spectra the same. We have noted that in the present approximation the energy input to very high wavenumbers comes principally from wavenumbers $p \sim k_s$. In the direct-interaction approximation, the local cascade which characterizes the inertial range persists throughout the dissipation range.

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[9] The definition of $\underset{\sim}{u}(\underset{\sim}{k})$ which yields (2.2) is

$$\underset{\sim}{u}(\underset{\sim}{k}) = (2\pi L)^{-3/2} \int \underset{\sim}{u}(\underset{\sim}{x}) \exp(-i\underset{\sim}{k}\cdot\underset{\sim}{x}) d^3x,$$

where the integration is over the whole box. This normalization is chosen because it gives

$$\frac{1}{2} \int |\underset{\sim}{u}(\underset{\sim}{k})|^2 d^3k = \int_0^\infty E(k)dk.$$

It should be noted that a strictly isotropic ensemble is possible only in the limit $L \to \infty$.

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[16] In this respect, the present approximation differs from the direct-interaction approximation, which yields a $k^{-3/2}$ law in the inertial range. See reference 7 for a discussion of the difference and its implications.

[17] A particularly simple form for computations would be

$$g(x) = N' \quad (c^{-1} < x < c)$$

$$= 0 \quad \text{(otherwise)},$$

where c is an arbitrary constant and N' is the normalization factor required by (4.4).

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